SOLUTION OF LAPLACE'S EQUATION IN INTERELECTRODE SPACE FOR SHAPED ELECTRODES*

Petr Novák, Arnošt Kimla, Ivo Roušar, Václav Cezner and Vladimír Mejta

Department of Inorganic Technology and Department of Mathematics, Prague Institute of Chemical Technology, 166 28 Prague 6

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The Laplace equation was solved in the interelectrode space for shaped electrodes (two-dimensional case) by the method of finite differences (FDM), Galerkin's method (GM), and collocation method (CM). A comparison shows that for electrode shapes with a continuously changing surface (continuous first and second derivatives), the solutions by all three methods are equivalent, giving identical distribution of local current densities on the electrode surface. The use of GM and CM is, however, not practical because of high requirements on the computer time and memory as compared with FDM. Moreover, the GM and CM fail in the case of discontinuities in the electrode shape.

The solution of Laplace's equation in an interelectrode space delimited by shaped electrodes is carried out by approximating the derivatives by difference formulas and using the method of finite differences in the given space[1,2]. In engineering problems, partial differential equations are often solved by variation methods[3]; which lead relatively rapidly to a sufficiently accurate result. For an interelectrode space delimited by shaped electrodes, Galerkin's method[4] using goniometric functions (series in terms of sin nx and cos nx) is suitable. The same functions can be used also in the collocation method. Our aim was to compare the individual methods from the point of view of accuracy of results, required time and memory of the computer.

MODEL SYSTEM

We shall solve Laplace's equation (*1*) in two-dimensional space delimited by electrodes whose form is given by curves $\alpha(x)$ and $\beta(x)$ (Fig. 1). The potentials in solution at the electrode surface, φ_A and φ_K, are given by functions $h_\alpha(x)$ and $h_\beta(x)$. The system is symmetrical or isolated in sections $x = a$ and $x = b$. Hence, the Laplace's equation has the form

$$\partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 = 0 \qquad (1)$$

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with boundary conditions for $\alpha(x)$ and $\beta(x)$:

$$\varphi_A(x) = h_\alpha(x)\,, \quad \varphi_K(x) = h_\beta(x)\,, \tag{2a,b}$$

$$(\partial\varphi/\partial x)_a = (\partial\varphi/\partial x)_b = 0\,. \tag{2c}$$

The functions α, β, h_α, and h_β are symmetrical in the points a and b:

$$\alpha'(a) = \alpha'(b) = \beta'(a) = \beta'(b) = h'_\alpha(a) = h'_\alpha(b) = h'_\beta(a) = h'_\beta(b) = 0\,. \tag{2d}$$

A transformation of the interelectrode space to a rectangle appears to be the simplest procedure. We use the following transformation:

$$\xi = x\,, \quad \eta = (y - \alpha(x))/(\beta(x) - \alpha(x))\,, \tag{3a,b}$$

$$\varphi(x, y) = \psi(\xi, \eta)\,. \tag{3c}$$

For the derivatives we then obtain

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial\xi} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial x} \tag{4a}$$

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\psi}{\partial\xi^2} + 2\frac{\partial^2\psi}{\partial\xi\,\partial\eta}\frac{\partial\eta}{\partial x} + \frac{\partial^2\psi}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{\partial\psi}{\partial\eta}\frac{\partial^2\eta}{\partial x^2}\,, \tag{4b}$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial y}\,, \quad \frac{\partial^2\varphi}{\partial y^2} = \frac{\partial^2\psi}{\partial\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2\,. \tag{4c,d}$$

Eq. (*1*) is now rewritten for the rectangle $a \leqq \xi \leqq b$, $0 \leqq \eta \leqq 1$ in the form:

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2}\left[\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2\right] + 2\frac{\partial^2\psi}{\partial\xi\,\partial\eta}\frac{\partial\eta}{\partial x} + \frac{\partial\psi}{\partial\eta}\frac{\partial^2\eta}{\partial x^2} = 0 \tag{5}$$

with boundary conditions

$$\eta = 0: \quad \psi(\xi, \eta) = h_\alpha(\xi)\,; \quad \eta = 1: \quad \psi(\xi, \eta) = h_\beta(\xi)\,, \tag{6a,b}$$

$$x = a\,, \quad x = b: \quad \partial\psi/\partial\xi + (\partial\psi/\partial\eta)\,\partial\eta/\partial x = 0\,. \tag{6c}$$

Owing to the symmetry, we have

$$(\partial\eta/\partial x)_a = (\partial\eta/\partial x)_b = 0\,, \tag{6d}$$

hence for $\xi = a$ and $\xi = b$:

$$\alpha'(\xi) = \beta'(\xi) = h'_\alpha(\xi) = h'_\beta(\xi) = 0\,. \tag{6e}$$

Eq. (5) can be rewritten in the form

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2}(\beta(x) - \alpha(x))^{-2}\,[1 + (\alpha'(x)(1-\eta) + \beta'(x)\,\eta)^2] -$$
$$- \frac{\partial^2\psi}{\partial\xi\,\partial\eta}\,\frac{2}{\beta(x)-\alpha(x)}\,[\alpha'(x)(1-\eta) + \beta'(x)\,\eta] - \frac{\partial\psi}{\partial\eta}(\beta(x) - \alpha(x))^{-2}\,.$$
$$.\,[(\beta(x) - \alpha(x))(1-\eta)\,\alpha''(x) + (\beta(x) - a(x))\,\eta\,\beta''(x) -$$
$$-2(\beta'(x) - \alpha'(x))(\alpha'(x)(1-\eta) + \beta'(x)\,\eta)] = 0\,. \tag{7}$$

Method of Finite Differences

The derivatives in Eq. (7) are replaced by difference formulas. Iterative calculation of potential values in the grid points is carried out by the relaxation method. After the calculation of the potentials in the space between the electrodes, the current densities on the electrode surface are calculated from Eqs (*22a*–*c*). The detailed procedure was described in ref.[1].

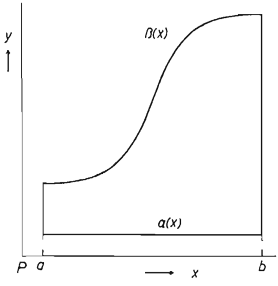

FIG. 1

Scheme of Interelectrode Space Limited by the Contours $\alpha(x)$ and $\beta(x)$.

Galerkin's Method

The basic idea consists in finding a function g, which fits the given boundary conditions, and then in introducing the function

$$w = \psi - g . \tag{8}$$

We define the following operator:

$$A = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right] + 2\,\frac{\partial^2}{\partial \xi\,\partial \eta}\,\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial \eta}\,\frac{\partial^2 \eta}{\partial x^2} \tag{9}$$

and thus we can write Eq. (*5*) with the use of (*8*) in the form

$$Aw = -Ag \tag{10}$$

with boundary conditions

$$w(\xi, 0) = 0 , \quad w(\xi, 1) = 0 \tag{11a,b}$$

and for

$$\xi = a \quad \text{or} \quad \xi = b : \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta}\,\frac{\partial \eta}{\partial x} = 0 . \tag{11c}$$

The latter equation is simplified by using the symmetry condition $\partial \eta / \partial x = 0$ to

$$(\partial w / \partial \xi)_{a,b} = 0 \tag{11d}$$

and the validity of (*6*) is preserved. The auxiliary function g must be chosen so as to fulfil the boundary conditions for ψ. In our case it is possible to use, for example, the function

$$g(\xi, \eta) = \eta\, h_\beta(\xi) + (1 - \eta)\, h_\alpha(\xi) \tag{12}$$

which for $\eta = 0$ fulfils the boundary condition (*2a*) and for $\eta = 1$ the boundary condition (*2b*). Further it follows from the symmetry condition (*2d*) that for $\xi = a$ and $\xi = b$ is $\partial g / \partial \xi = 0$, whereby the boundary condition (*2c*) is fulfilled.

The solution for w will be sought in the form

$$w(\xi, \eta) = \sum_{k=1}^{N} \sum_{l=1}^{N} a_{k,l} c_{k,l}(\xi, \eta) , \tag{13}$$

where $c_{k,l}(\xi, \eta)$ are the so-called base functions which must identically fulfil the boundary conditions (*11a,b*), and $a_{k,l}$ are coefficient to be determined. The base functions can be, for example, chosen as products of goniometric functions:

$$c_{k,l}(\xi, \eta) = \cos\left(\frac{\xi - a}{b - a}(k - 1)\,\pi\right)\sin\left(\pi\, l\eta\right), \tag{14}$$

where $k, l = 1, 2, 3 \ldots N$. The total number of unknown coefficients $a_{k,l}$ is N^2. These are determined according to the Galerkin's method so that the approximation of w, Eq. (*13*), is substituted into the partial differential equation (*10*) and this is scalarially multiplied in turn by the functions $c_{m,n}(\xi, \eta)$, $m, n = 1, 2, \ldots, N$. Thus, we obtain N^2 equations for N^2 unknowns $a_{k,l}$. This system of equations can be written in the form

$$\sum_{k=1}^{N} \sum_{l=1}^{N} a_{k,l}(Ac_{k,l};\, c_{m,n}) = -(Ag;\, c_{m,n}), \tag{15}$$

where $m, n = 1, 2, \ldots, N$. The scalar product on the left side has the form

$$\int_{\xi=a}^{\xi=b} \int_{\eta=0}^{\eta=1} (Ac_{k,l})\, c_{m,n}\, d\xi\, d\eta \tag{16}$$

and the scalar product on the right side of (*15*) can be written analogously. Owing to a suitable choice of the base functions, integration of Eq. (*16*) with respect to η can be relatively easily carried out. The method of calculating the scalar products on the left side of Eq. (*15*) is as follows:

a) Calculation of coefficients in differential operator A.

$$\frac{\partial \eta}{\partial x} = -\frac{\alpha'(x) - \eta(\beta'(x) - \alpha'(x))}{\beta(x) - \alpha(x)}, \tag{16a}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\beta(x) - \alpha(x)}, \tag{16b}$$

$$\frac{\partial^2 \eta}{\partial x^2} = -\frac{\alpha''(x)}{\beta(x) - \alpha(x)} + 2\alpha'(x)\frac{\beta'(x) - \alpha'(x)}{(\beta(x) - \alpha(x))^2} + \\ + \eta\left[2\left(\frac{\beta'(x) - \alpha'(x)}{\beta(x) - \alpha(x)}\right)^2 - \frac{\beta''(x) - \alpha''(x)}{\beta(x) - \alpha(x)}\right] \tag{16c}$$

$$\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 = (\beta - \alpha)^{-2}\left[1 + (\alpha'(1 - \eta) + \beta'\eta)^2\right]. \tag{16d}$$

Now the operator A can be expressed in the form

$$A = \frac{\partial^2}{\partial \xi^2} + (p_1(\xi)\,\eta + p_2(\xi)\,\eta^2 + p_3(\xi))\,\frac{\partial^2}{\partial \eta^2} +$$

$$+ 2(q_1(\xi)\,\eta + q_2(\xi))\,\frac{\partial^2}{\partial \xi\,\partial \eta} + (r_1(\xi)\,\eta + r_2(\xi))\,\frac{\partial}{\partial \eta}\,, \tag{17a}$$

where

$$p_1(\xi) = 2\alpha'\,\frac{\beta' - \alpha'}{(\beta - \alpha)^2}\,, \quad p_2(\xi) = \left(\frac{\beta' - \alpha'}{\beta - \alpha}\right)^2\,, \tag{17b,c}$$

$$p_3(\xi) = \frac{1 + (\alpha')^2}{(\beta - \alpha)^2}\,, \tag{17d}$$

$$q_1(\xi) = -\,\frac{\beta' - \alpha'}{\beta - \alpha}\,, \quad q_2(\xi) = -\,\frac{\alpha'}{\beta - \alpha}\,, \tag{17e,f}$$

$$r_1(\xi) = 2\left(\frac{\beta' - \alpha'}{\beta - \alpha^2}\right)^2 - \frac{\beta'' - \alpha''}{\beta - \alpha}\,, \tag{17g}$$

$$r_2(\xi) = -\,\frac{\alpha''}{\beta - \alpha} + 2\alpha'\,\frac{\beta' - \alpha'}{(\beta - \alpha)^2}\,. \tag{17h}$$

By substituting Eq. (*14*) into the formula for A we obtain:

$$Ac_{k,l}(\xi, \eta) = -\,\frac{(k - 1)^2\,\pi^2}{(b - a)^2}\cos\left(\frac{\xi - a}{b - a}(k - 1)\,\pi\right)\sin(l\eta\pi) -$$

$$-\,[p_1(\xi)\,\eta + p_2(\xi)\,\eta^2 + p_3(\xi)]\,\pi^2 l^2\cos\left(\frac{\xi - a}{b - a}(k - 1)\,\pi\right)\sin(l\eta\pi) -$$

$$-\,2[q_1(\xi)\,\eta + q_2(\xi)]\,\frac{(k - 1)\,l\pi^2}{b - a}\sin\left(\frac{\xi - a}{b - a}(k - 1)\,\pi\right)\cos(l\eta\pi) +$$

$$+\,[r_1(\xi)\,\eta + r_2(\xi)]\,l\pi\cos\left(\frac{\xi - a}{b - a}(k - 1)\,\pi\right)\cos(l\eta\pi)\,. \tag{18}$$

Similarly by substituting (*12*) into (*17a*) we obtain

$$Ag(\xi, \eta) = h''_\alpha(\xi) + 2q_2(\xi)\,(h'_\beta(\xi) - h'_\alpha(\xi)) + r_2(\xi)\,(h_\beta(\xi) - h_\alpha(\xi)) +$$
$$+ \eta(h''_\beta(\xi) - h''_\alpha(\xi)) + 2q_1(\xi)\,[h'_\beta(\xi) - h'_\alpha(\xi) + r_1(\xi)\,(h_\beta(\xi) - h_\alpha(\xi))]\,. \quad (19)$$

The following expression for the scalar product $(Ac_{k,l};\, c_{m,n})$ can be obtained by substituting Eq. (*18*) into (*16*):

$$(Ac_{k,l};\, c_{m,n}) = -\frac{(k-1)^2\,\pi^2}{4(b-a)}\,\delta_{k,m}\delta_{l,n} + \int_a^b \left\{-\frac{l^2\pi^2}{2}\,\delta_{l,n}\,p_3(\xi) + \right.$$
$$+ \tfrac{1}{2}l^2(1-\delta_{l,n})\left[\frac{\cos(\pi(l+n)) - 1}{(l+n)^2} - \frac{\cos(\pi(l-n)) - 1}{(l-n)^2}\right] p_1(\xi) - \tfrac{1}{4}l^2\delta_{l,n}\pi^2\,p_1(\xi) +$$
$$+ l^2(1-\delta_{l,n})\left[\frac{\cos(\pi(l+n))}{(l+n)^2} - \frac{\cos((l-n)\,\pi)}{(l-n)^2}\right] p_2(\xi) + l^2\delta_{l,n}\left(\frac{1}{4l^2} - \frac{\pi^2}{6}\right) p_2(\xi) -$$
$$- \tfrac{1}{2}(1-\delta_{l,n})\left[\frac{\cos(\pi(l+n)) - 1}{l+n} - \frac{\cos(\pi(l-n)) - 1}{l-n}\right] r_2(\xi) - \tfrac{1}{2}(1-\delta_{l,n})\,.$$
$$.\left[\frac{\cos(\pi(l+n))}{l+n} - \frac{\cos(\pi(l-n))}{l-n}\right] r_1(\xi) - \tfrac{1}{4}\delta_{l,n}\,r_1(\xi)\Bigg\}\cos\left(\frac{\xi-a}{b-a}(k-1)\,\pi\right).$$
$$.\cos\left(\frac{\xi-a}{b-a}(m-1)\,\pi\right)\mathrm{d}\xi + \int_a^b \left\{\frac{(k-1)\,l\pi}{b-a}(1-\delta_{l,n})\left(\frac{\cos((l+n)\pi) - 1}{l+n} - \right.\right.$$
$$\left. - \frac{\cos((l-n)\,\pi) - 1}{l-n}\right) q_2(\xi) + \frac{(k-1)\,l\pi}{b-a}(1-\delta_{l,n})\left(\frac{\cos((l+n)\pi)}{l+n} - \right.$$
$$\left.\left. - \frac{\cos((l-n)\,\pi)}{l-n}\right) q_1(\xi) + \frac{(k-1)\,\pi}{2(b-a)}\,\delta_{l,n}\,q_1(\xi)\right\}\sin\left(\frac{\xi-a}{b-a}(k-1)\,\pi\right).$$
$$.\cos\left(\frac{\xi-a}{b-a}(m-1)\,\pi\right)\mathrm{d}\xi\,. \quad (20)$$

The Kronecker's symbol $\delta_{i,j}$ is defined as follows: $\delta_{i,j} = 0$ for $i \neq j$, $\delta_{i,j} = 1$ for $i = j$. From Eq. (*19*) we obtain in a similar way expressions for the scalar products $(Ag;\, c_{m,n})$:

$$(Ag;\, c_{m,n}) = \frac{1 - \cos(n\pi)}{n\pi} \int_a^b \{h''_\alpha(\xi) + 2q_2(\xi)[h'_\beta(\xi) - h'_\alpha(\xi)] + r_2(\xi)\,[h_\beta(\xi) - h_\alpha(\xi)]\} \,.$$

$$. \cos\left(\frac{\xi - a}{b - a}(m - 1)\,\pi\right) \mathrm{d}\xi - \frac{\cos(n\pi)}{n\pi} \int_a^b \{h''_\beta(\xi) - h'_\alpha(_D\xi) +$$

$$+ 2q_1(\xi)\,[h'_\beta(\xi) - h'_\alpha(\xi)] + r_1(\xi)\,[h_\beta(\xi) - h_\alpha(\xi)]\} \cos\left(\frac{\xi - a}{b - a}(m - 1)\,\pi\right) \mathrm{d}\xi \,. \quad (21)$$

Integration in (*20*) and (*21*) is carried out numerically; the integration step h must be by an order of magnitude smaller than the smallest period: $h \ll (b - a)/N$. By substituting Eqs (*20*) ad (*21*) into (*15*) we obtain in total N^2 equations for N^2 unknown coefficients $a_{k,l}$. The solution of this system of equations by an arbitrary finite or iteration method leads to the sought value of $a_{k,l}$.

The current densities on the electrode surface are calculated from the equation

$$i_n = -\varkappa_E\,(\mathrm{grad}\;\varphi)_n\,, \qquad (22a)$$

hence for an electrode whose shape is given by the curve $\alpha(\xi)$:

$$(\mathrm{grad}\;\varphi)_{n,\alpha} = (1 + \alpha'^2)^{-1/2} \left[\left(\frac{\partial\psi}{\partial\eta}\right)_\alpha \frac{1 + \alpha'^2}{\beta - \alpha} - \left(\frac{\partial\psi}{\partial\xi}\right)_\alpha \alpha'\right] \qquad (22b)$$

and for the other electrode of the shape given by $\beta(\xi)$:

$$(\mathrm{grad}\;\varphi)_{n,\beta} = (1 + \beta'^2)^{-1/2} \left[\left(\frac{\partial\psi}{\partial\eta}\right)_\beta \frac{1 + \beta'^2}{\beta - \alpha} - \left(\frac{\partial\psi}{\partial\xi}\right)_\beta \beta'\right]. \qquad (22c)$$

The latter two equations give the values of normal components of the gradients except for their sign. Provision must be made that the anodic current density be positive and the cathodic one negative.

Collocation Method

In this method, all unknown functions are approximated by suitable functional relations with unknown multiplicative coefficients. The solution of the corresponding differential equation is then reduced to the solution of a system of linear equations for the mentioned coefficients. In solving the Laplace's equation, we start from Eqs (*9*), (*10*), (*12*), and (*13*) with the corresponding boundary conditions. Approximations for $c_{k,l}$ are chosen in the form (*14*); by differentiating we obtain

$$\frac{\partial c_{k,l}}{\partial \eta} = \pi l \cos \left[\frac{\xi - a}{b - a}(k - 1)\pi\right] \cos(l\eta\pi), \tag{23a}$$

$$\frac{\partial^2 c_{k,l}}{\partial \eta^2} = -\pi^2 l^2 \cos \left[\frac{\xi - a}{b - a}(k - 1)\pi\right] \sin(l\eta\pi). \tag{23b}$$

$$\frac{\partial^2 c_{k,l}}{\partial \eta\, \partial \xi} = \frac{-\pi^2 l(k - 1)}{b - a} \sin \left[\frac{\xi - a}{b - a}(k - 1)\pi\right] \cos(l\eta\pi), \tag{23c}$$

$$\frac{\partial^2 c_{k,l}}{\partial \xi^2} = -\frac{(k - 1)\pi^2}{(b - a)^2} \cos \left[\frac{\xi - a}{b - a}(k - 1)\pi\right] \sin(l\eta\pi). \tag{23d}$$

By substituting these expressions into (*10*) and using (*13*) we obtain N^2 linear equations for N^2 multiplicative constants $a_{k,l}$. This system of equations will be solved by the Gauss' method (the method of solution is arbitrary).

The number of collocation points in the direction of ξ or η is equal to N. The collocation points will be defined by the conditions

$$\cos\left(\frac{\xi - a}{b - a} N\pi\right) = 0 \quad \text{for} \quad a \leqq \xi \leqq b\,, \quad \sin\left[(N + 1)\eta\pi\right] = 0 \quad \text{for} \quad 0 \leqq \eta \leqq 1\,, \tag{24a,b}$$

representing zero points of approximation functions whose order is by 1 higher $(N + 1)$ than for the functions used in the approximation. It follows from the theory that with this choice of the collocation points the solution will be most accurate in the sense of least squares of deviations between the exact and approximate solutions. Since Eq. (*24b*) has $N + 2$ solutions, we drop two arbitrary points. To involve the boundaries $\xi = a$ and $\xi = b$ in the calculation, we replace two arbitrary collocation points from the solution of Eq. (*24a*) by points for $\xi = a$ and $\xi = b$. The current densities on the electrode surface are calculated from (*22a–c*); the necessary values of the derivatives are obtained from Eq. (*8*) by combining with (*13*) and (*14*) and rearranging:

$$\frac{\partial \psi}{\partial \xi} = \eta\, h'_\beta(\xi) + (1 - \eta)\, h'_\alpha(\xi) \tag{25a}$$

$$\frac{\partial \psi}{\partial \eta} = h_\beta(\xi) - h_\alpha(\xi) + \sum_{k=1}^{N} \sum_{l=1}^{N} a_{k,l} \pi l \cos\left(\frac{\xi - a}{b - a}(k - 1)\pi\right) \cos(l\eta\pi). \tag{25b}$$

RESULTS AND DISCUSSION

Calculations were carried out for a two-dimensional electrolyser whose one electrode shape corresponds to the function $\beta(x) = k_1 - k_2 \cos[(x - a)\pi/(b - a)]$ and the second to $\alpha(x) = 0$. This was chosen because of the continuity of both functions and their derivatives and because the cell form resembles the Hull's cell. We considered the primary current distribution, hence constant potentials on both electrodes, $h_\beta(x) = 1$ and $h_\alpha(x) = 0$. The current densities at $\beta(x)$ are positive (anode) and at $\alpha(x)$ negative (cathode).

In Table I are shown the results for current densities in different points on the electrodes calculated with the three methods for $\beta(x) = 2{\cdot}4 - 1{\cdot}8\cos(\pi X/3{\cdot}6)$, where $X = x - a$. In Table II analogously for $\beta(x) = 6{\cdot}0 - 5{\cdot}4\cos(\pi X/3{\cdot}6)$. In the case of FDM we used 7 × 7 and 19 × 19 points, with GM 7 × 7 functions, and with CM 7×7 and 11×11 functions. We assumed $\varkappa_E = 1\,\Omega^{-1}\,cm^{-1}$. The computer was of the type ICL 4−72. It is seen that in the first case all three methods give practically the same results. In the second case, where the electrode was more steep, only the FDM gives acceptable results. This method is especially suitable for calculations where the polarization is taken into account, since the computer time increases

TABLE I

Values of Current Densities on the Surface of Shaped Anode ($\beta(x) = 2{\cdot}4$—$1{\cdot}8\cos(\pi X/3{\cdot}6)$) and Cathode ($\alpha(x) = 0$) Calculated by Different Methods for Different N Values

x	$\beta(x)$	FDM				GM		CM			
		$N = 19$		$N = 7$		$N = 7$		$N = 11$		$N = 7$	
		$i_{N,\beta}$	$-i_{N,\alpha}$	$i_{N,\beta}$	$-i_{N,\alpha}$	$i_{N,\beta}$	$-i_{N,\alpha}$	$i_{N,\beta}$	$-i_{N,\alpha}$	$i_{N,\beta}$	$-i_{N,\alpha}$
0·0	0·60	2·071	1·509	2·061	1·496	2·013	1·508	2·019	1·510	1·996	1·510
0·6	0·84	1·293	1·226	1·331	1·221	1·196	1·173	1·323	1·227	1·335	1·225
1·2	1·50	0·603	0·842	0·607	0·838	0·579	0·809	0·694	0·844	0·762	0·833
1·8	2·40	0·294	0·611	0·283	0·610	0·305	0·615	0·314	0·613	0·316	0·608
2·4	3·30	0·139	0·486	0·130	0·490	0·171	0·502	0·152	0·489	0·127	0·490
3·0	3·96	0·065	0·425	0·059	0·431	0·076	0·434	0·075	0·428	0·076	0·429
3·6	4·20	0·043	0·410	0·038	0·413	0·052	0·409	0·050	0·409	0·052	0·411
Comp. time s		270		47		266		79		7	
Comp. memory KBYTE		67		61		94		163		64	

only by 10% (ref.[1]). With GM and CM, if the polarization were taken into account, the whole calculation would have to be repeated for several times and thus the computer time would increase by an order of magnitude. The accuracy of GM and CM could be increased by increasing substantially the number of functions, however this would result in extreme requirements regarding the computer time and memory.

The calculation was further carried out also for $\beta(x) = 0{\cdot}6 + 0{\cdot}5x$, *i.e.*, for an electrode which has in the points $x = a$ and $x = b$ undefined derivatives $\beta'(x)$ and $\beta''(x)$. Acceptable results were obtained only with FDM, whereas with GM and CM oscillations of the potential and current density values took place owing to discontinuities on the boundaries.

CONCLUSIONS

In the case of electrode shapes with continuous first and second derivatives in all points of the surface, the three methods, FDM, GM, and CM are equivalent with respect to the obtained results. In other cases ony the finite difference method can be

TABLE II

Values of Current Densities on the Surface of Shaped Anode ($\beta(x) = 6{\cdot}0 - 5{\cdot}4 \cos(\pi X/3{\cdot}6)$) and Cathode ($\alpha(x) = 0$) Calculated by Different Methods and for Different N Values

		FDM				GM		CM			
x	$\beta(x)$	$N = 19$		$N = 7$		$N = 7$		$N = 11$		$N = 7$	
		$i_{N,\beta}$	$-i_{N,\alpha}$	$i_{N,\beta}$	$-i_{N,\alpha}$	$i_{N,\beta}$	$-i_{N,\alpha}$	$i_{N,\beta}$	$-i_{N,\alpha}$	$i_{N,\beta}$	$-i_{N,\alpha}$
0·0	0·60	2·698	1·381	2·912	1·266	2·136	1·408	2·475	1·385	2·425	1·383
0·6	1·32	0·660	1·035	0·701	1·037	0·432	0·973	1·070	1·033	0·100	1·026
1·2	3·30	0·157	0·668	0·113	0·671	0·259	0·642	0·665	0·657	3·791	0·581
1·8	6·00	0·025	0·486	—0·005	0·497	—0·059	0·489	0·190	0·481	—0·205	0·412
2·4	8·70	0·001	0·394	—0·016	0·411	0·073	0·403	—0·213	0·392	—0·392	0·339
3·0	10·68	0·000	0·350	—0·005	0·370	—0·048	0·353	—0·007	0·346	—0·044	0·296
3·6	11·40	0·000	0·337	—0·001	0·358	0·021	0·332	—0·001	0·333	—0·004	0·284
Comp. time s		360		47		266		79		7	
Comp. memory KBYTE		67		61		94		163		64	

successfully used. The latter has also the least requirements on the computer memory and, if we consider the accuracy of the results, also on the computer time. At a constant number of points in the grid, the computer time becomes longer with increasing values of $\beta'(x)$ and $\beta''(x)$ (Tables I and II, results for $N = 19$). The collocation method turned out to be most rapid one (with constant number of functions), however the results were worst: the current densities on the electrode surface $\beta(x)$ oscillated markedly and only after increasing the number of functions to 11 the results were acceptable in the region where the electrodes were relatively close to each other. Nevertheless, occasionally an opposite sign of the current density was obtained, which did not correspond to the respective electrode and hence had no physical sense. With the GM, a substantial part of the computer time is consumed in numerical integration of Eqs (*20*) and (*21*). Therefore, only the finite difference method is recommended for similar calculations.

REFERENCES

1. Novák P., Roušar I., Kimla A., Cezner V., Mejta V.: This Journal, in press.
2. McGeough J. A.: *Principles of Electrochemical Machining*, p. 227. Chapman and Hall, London 1974.
3. Rektorys K.: *Variační metody v inženýrských problémech a problémech matematické fysiky*. Published by SNTL, Prague 1974.
4. Ref. 3, p. 170.

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